

ON THE UNIFORM CYCLE-ROOTED SPANNING TREE IN \mathbb{Z}^2

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ABSTRACT. We compute the asymptotics of the first and second moments of the area of the cycle of a random cycle-rooted spanning tree (spanning unicycle) of any sequence of graphs $\mathcal{G}_n \subset \mathbb{Z}^2$, such that $\frac{1}{n}\mathcal{G}_n$ approximates a bounded domain $D \subset \mathbb{C}$. We show that the first and second moments grow like $\frac{4}{\pi} \log n$ and $C \cdot \text{Area}(D)n^2$, respectively, for an explicit constant $C = C(D)$.

We use these results to give a lower bound for the first and third moments of the length of the random loop obtained by adding an independent random edge to a uniform spanning tree on \mathcal{G}_n .

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1. INTRODUCTION

A *cycle-rooted spanning tree* (CRST), also called spanning unicycle, on a graph \mathcal{G} is a connected spanning subgraph containing a unique cycle (in other words a connected subgraph with as many edges as vertices). More generally a *cycle-rooted spanning forest* (CRSF) is a subgraph each of whose connected components has a unique cycle, that is, is a cycle-rooted tree.

On any connected finite planar graph \mathcal{G} we denote by $\mathbf{\Gamma}_{\mathcal{G}}$ the random variable corresponding to the uniform measure on CRSTs. We let $\mathbf{L}_{\mathcal{G}}$ and $\mathbf{A}_{\mathcal{G}}$ be the combinatorial length (number of edges) and the combinatorial area (number of faces) of the cycle of $\mathbf{\Gamma}_{\mathcal{G}}$.

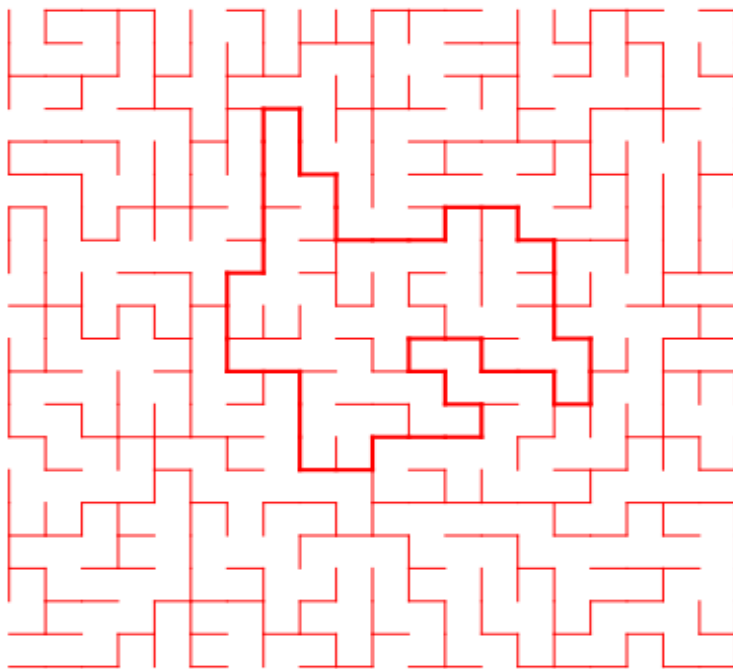


FIGURE 1. A sample of a uniform CRST on a 21×21 square grid conditioned on having a longish cycle.

We show (Theorems 3 and 4) that for the $n \times n$ square grid \mathcal{G} the area satisfies $\mathbb{E}(\mathbf{A}_{\mathcal{G}}) = \frac{4}{\pi} \log(n) + o(\log(n))$ and $\mathbb{E}(\mathbf{A}_{\mathcal{G}}^2) = Cn^2 + o(n^2)$ where $C = \frac{512\beta}{\pi^6} \approx .281$ and β is the constant

$$\beta = \sum_{k,l=1, \text{ both odd}}^{\infty} \frac{1}{k^2 l^2 (k^2 + l^2)} \approx .527.$$

We extend this result (Theorem 8) to any exhaustion \mathcal{G}_n of \mathbb{Z}^2 such that $\frac{1}{n}\mathcal{G}_n$ approximates a bounded domain $D \subset \mathbb{C}$. We show that the expected area satisfies the same asymptotics, that is $\mathbb{E}(\mathbf{A}_{\mathcal{G}_n}) = \frac{4}{\pi} \log n + o(\log n)$, and that $\mathbb{E}(\mathbf{A}_{\mathcal{G}_n}^2)$ grows like $C(D)|D|n^2$, where $|D|$ is the area of D and

$$C(D) = \frac{8}{|D|^2} \int_{D^2} g_D^0(z, w) |dz|^2 |dw|^2,$$

depends on D as the integral of its Dirichlet Green's function g_D^0 . We call $C(D)$ the *mean normalized exit time* of D , since it is (when multiplied by $|D|/4$) the mean exit time for Brownian motion started from a uniform random point; it is invariant under similarities of D . For an $n \times m$ rectangle with aspect ratio $m/n \rightarrow \tau$, the mean normalized exit time is $C = \frac{512\beta(\tau)}{\pi^6}$, where

$$\beta(\tau) = \sum_{k,l=1, \text{ both odd}}^{\infty} \frac{\tau}{k^2 l^2 (k^2 + \tau^2 l^2)}.$$

For the disk the mean normalized exit time is $C = 1/\pi$, see below. This is the maximal possible value of $C(D)$ among all Jordan domains. The constant $C(D)$ also has a mechanical interpretation as $C(D) = \frac{2}{|D|^2} P(D)$, where $P(D)$ is the torsional rigidity of a cylinder beam with cross-section D , see Section 5.

The uniform CRST is closely related to the uniform spanning tree, but on general graphs no formula is known even for the total number $\lambda_{\mathcal{G}}$ of CRSTs. On increasing exhaustions of the square grid $(\mathcal{G}_n)_{n \geq 1}$ the number of CRSTs was recently shown by Levine and Peres [12] (based on results of Kenyon and Wilson [9]) to be $\frac{|V_n|}{8} \kappa_{\mathcal{G}_n} (1 + o(1))$, where $|V_n|$ is the number of vertices of \mathcal{G}_n and $\kappa_{\mathcal{G}_n}$ is the number of spanning trees of \mathcal{G}_n . As a simple corollary they showed that the expected length of the cycle in a uniform CRST on the grid is $8 + o(1)$. A similar argument shows that $\mathbb{E}(\mathbf{L}_{\mathcal{G}_n}) = 8 + o(1)$, when $\frac{1}{n}\mathcal{G}_n$ approximates any bounded domain $D \subset \mathbb{C}$.

The unusual feature of this random cycle, that the average length is finite while the average area grows, is a consequence of the “long tail” of the distribution. It is natural to conjecture that the probability that the area is K decays like K^{-2} , and (based on the known relation between length and diameter for the branches of the uniform spanning tree [7]) the length L and area K are related by $L = O(K^{5/8})$.

Using the techniques of [1], one can explicitly compute the probability that the cycle has any given shape (in particular any given length), in the limit $n \rightarrow \infty$, since removing a fixed edge from the cycle leaves a uniform spanning tree containing the remaining set of edges. The probability of lengths 4, 6, 8 are respectively

$$-\frac{16}{\pi^3} + \frac{8}{\pi^2} \approx .294$$

$$\begin{aligned} & \frac{256}{\pi^4} - \frac{352}{\pi^3} + \frac{144}{\pi^2} - \frac{18}{\pi} \approx .136 \\ & -\frac{83886080}{81\pi^7} + \frac{160890880}{81\pi^6} - \frac{128954368}{81\pi^5} + \frac{56350208}{81\pi^4} - \frac{4886768}{27\pi^3} + \frac{85100}{3\pi^2} - \frac{7429}{3\pi} + \frac{355}{4} \approx .071. \end{aligned}$$

2. LAPLACIAN, GREEN'S FUNCTION, AND TRANSFER IMPEDANCE

Let \mathcal{G} be a finite graph with vertex set V . We let $\kappa_{\mathcal{G}}$ be the number of spanning trees and $\lambda_{\mathcal{G}}$ the number of CRSTs of \mathcal{G} . For any edge e (respectively any face f), we write $e \in \mathbf{\Gamma}_{\mathcal{G}}$ (respectively $f \in \mathbf{\Gamma}_{\mathcal{G}}$) to express the event that the edge is part of the cycle of $\mathbf{\Gamma}_{\mathcal{G}}$ (respectively, that f is in the interior of the cycle of $\mathbf{\Gamma}_{\mathcal{G}}$).

The main tool for computing the quantities we are interested in will be the transfer impedance. Let us recall some basic facts about it and its link with the Green's function.

We define the Laplacian $\Delta : \mathbb{R}^V \rightarrow \mathbb{R}^V$ by $\Delta f(v) = \sum_{v' \sim v} f(v') - f(v)$, the sum being over nearest neighbors v' of v . As long as \mathcal{G} is connected Δ has kernel consisting of the constant functions. Δ is invertible on the orthocomplement V_0 of the constant functions and we define the (Neumann) Green's function G to be this inverse. By abuse of notation for vertices a, b, x we denote $G(a, x) - G(b, x)$ the function of x defined by $\Delta^{-1}(\delta_a - \delta_b)$.

Let $\Delta_{a,b}^{c,d}$ be the determinant of the submatrix of the Laplacian in which rows a, b and columns c, d have been deleted.

Lemma 1 (See e.g. [13]).

$$(1) \quad (-1)^{a+b+c+d} \Delta_{a,b}^{c,d} = \kappa_{\mathcal{G}} (G(a, c) - G(b, c) + G(b, d) - G(a, d)).$$

Given two oriented edges $e_1 = ab$ and $e_2 = cd$, define the *transfer impedance* $T_{\mathcal{G}}(e_1, e_2)$ by $T(ab, cd) = G(a, c) - G(b, c) - G(a, d) + G(b, d)$. The transfer impedance has a random walk interpretation and an equivalent electrical interpretation, see [1], [3], or [13].

Lemma 2 (See e.g. [13]). *If we interpret the graph as a resistor network with unit resistances on the edges, then $T(ab, cd)$ is the amount of current crossing edge cd when one unit of current enters at a and leaves at b .*

One can give an expression for T involving an eigenbasis of the Laplacian. We denote by $\langle \cdot, \cdot \rangle$ the usual scalar product on V_0 , coming from restriction of the standard scalar product on \mathbb{R}^V . Let $\{f_k\}$ be an orthonormal basis of V_0 consisting of eigenvectors for Δ associated to eigenvalues $\lambda_k \neq 0$. Then, for any two functions $f, g \in V_0$, we have

$$\langle f, \Delta_{|V_0}^{-1} g \rangle = \sum_k \frac{\langle f_k, f \rangle \langle f_k, g \rangle}{\lambda_k},$$

since this equality holds when f and g are replaced by any elements of the basis $\{f_k\}$.

Therefore we may write the transfer impedance as

$$\begin{aligned}
T(ab, cd) &= \langle \delta_c - \delta_d, \Delta^{-1}(\delta_a - \delta_b) \rangle \\
&= \sum_k \frac{\langle f_k, \delta_c - \delta_d \rangle \langle f_k, \delta_a - \delta_b \rangle}{\lambda_k} \\
(2) \qquad &= \sum_k \frac{(f_k(c) - f_k(d))(f_k(a) - f_k(b))}{\lambda_k}.
\end{aligned}$$

Recall also the Matrix-Tree theorem

Theorem 1 ([10]). *For any row a and column b , $\kappa_{\mathcal{G}} = (-1)^{a+b} \Delta_a^b$.*

3. ESTIMATES FOR GENERAL FINITE GRAPHS

3.1. Moments. The general strategy for computing moments of $\mathbf{A}_{\mathcal{G}}$ is to use the following simple observation.

Lemma 3. *For any integer $k \geq 1$,*

$$\mathbb{E}(\mathbf{A}_{\mathcal{G}}^k) = \sum_{f_1, \dots, f_k} \mathbb{P}(f_1, \dots, f_k \in \Gamma_{\mathcal{G}})$$

where the sum is over all ordered k -tuples of bounded faces.

In this section, we use the line-bundle Laplacian [8] to derive the probabilities $\mathbb{P}(f_1, \dots, f_k \in \Gamma_{\mathcal{G}})$ for $k = 1$ and 2. Let us briefly recall the definition of this Laplacian and the result relating its determinant to the CRSFs of the graph.

3.2. Line-bundle Laplacian. Let $\mathcal{G} = (V, E)$ be a finite graph. A \mathbb{C} -*bundle* is a copy \mathbb{C}_v of \mathbb{C} associated to each vertex $v \in V$. The *total space* of the bundle is the direct sum $W = \bigoplus_{v \in V} \mathbb{C}_v$. A *connection* Φ on W is the data consisting of, for each edge $e = vv'$, a complex linear isomorphism $\varphi_{vv'} : \mathbb{C}_v \rightarrow \mathbb{C}_{v'}$. That is, we assign to each oriented edge e a non-zero complex number φ_e such that $\varphi_{-e} = \varphi_e^{-1}$. We say that φ_e is the *parallel transport* of the connection over edge e . We say that two connections Φ, Φ' are *gauge equivalent* if there exist non-zero complex numbers ψ_v such that $\psi_{v'} \varphi_{vv'} = \varphi'_{vv'} \psi_v$, that is, Φ' is obtained from Φ by changing the basis of the \mathbb{C}_v .

The *monodromy* of a connection around an oriented cycle $v_1, \dots, v_{n+1} = v_1$ is the complex number $\prod_{i=1}^n \varphi_{v_{i+1}, v_i} \in \mathbb{C}^*$.

We let $\Delta = \Delta_\Phi$ be the associated Laplacian acting on $f \in W$ defined by, for each vertex v

$$\Delta(f)(v) = \sum_{v' \sim v} f(v) - \varphi_{v'v} f(v'),$$

where the sum is over neighbors v' of v .

Theorem 2 ([4, 8]). *We have*

$$\det(\Delta) = \sum_{\text{CRSFs cycles}} \prod (2 - \omega - 1/\omega),$$

where ω is the monodromy of the connection around the cycle, for either choice of its orientation.

3.3. Two technical lemmas. We first start with two technical lemmas.

Lemma 4. *Let M be a square matrix and Z a block-diagonal matrix of the same size:*

$$Z = \begin{pmatrix} B_1 & & & & & \\ & B_2 & & & & \\ & & \ddots & & & \\ & & & B_k & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

consisting of blocks of the form $B_i = \begin{pmatrix} 0 & p_i \\ q_i & 0 \end{pmatrix}$, where $\{p_i, q_i\}_{1 \leq i \leq k}$ are $2k$ variables, and 0 elsewhere. We denote by x_i, y_i the two consecutive rows (and columns) indices corresponding to the i -th block. Let $F := F_M(p_1, \dots, p_k, q_1, \dots, q_k)$ be the polynomial $\det(M + Z) \in \mathbb{C}[\{p_i, q_i\}_{1 \leq i \leq k}]$. We denote by H the orthogonal projection of F on the space of polynomials of total degree at most 2. Then

$$H = \sum_{i=1}^k (-q_i M_{x_i}^{y_i} - p_i M_{y_i}^{x_i} - p_i q_i M_{x_i, y_i}^{x_i, y_i}) \\ + \sum_{1 \leq i < j \leq k} (p_i p_j M_{y_i, y_j}^{x_i, x_j} + q_i q_j M_{x_i, x_j}^{y_i, y_j} + p_i q_j M_{y_i, x_j}^{x_i, y_j} + q_i p_j M_{x_i, y_j}^{y_i, x_j}),$$

where M_A^B denotes the determinant of the submatrix of M where the rows indexed by A and the columns indexed by B are removed.

Proof. We compute explicitly the coefficient of each of the monomials that can appear in H : $p_i, q_i, p_i p_j, q_i q_j, p_i q_i, p_i q_j$, and $q_i p_j$, for $i, j = 1, \dots, k$ and $i < j$.

When expanding the determinant along the y_i -th row, the term p_i appears only once, and in order to get the degree 1 term, we set p_j, q_j to be zero in the corresponding minor. This yields that the coefficient of p_i in F is $F[[p_i]] = -M_{y_i}^{x_i}$. Similarly, expanding along the y_i -th column, we obtain that $F[[q_i]] = -M_{x_i}^{y_i}$.

We also obtain that for all i , $F[[p_i q_i]] = -M_{x_i, y_i}^{x_i, y_i}$. And for $i < j$, we have $F[[p_i p_j]] = M_{y_i, y_j}^{x_i, x_j}$, and $F[[q_i q_j]] = M_{x_i, x_j}^{y_i, y_j}$, and $F[[p_i q_j]] = M_{y_i, x_j}^{x_i, y_j}$, and $F[[q_i p_j]] = M_{x_i, y_j}^{y_i, x_j}$. The result follows by summing all the terms. \square

The next lemma is a generalization of the previous one to a degenerate setting.

Lemma 5. *Let M be a square matrix and $\{p_i, q_i\}_{1 \leq i \leq k}$ be $2k$ parameters. Let Z be a block-diagonal matrix of the same size as M with blocks of the form $\begin{pmatrix} 0 & p_i & 0 \\ q_i & 0 & p_i \\ 0 & q_i & 0 \end{pmatrix}$*

for $i \leq r$, and $\begin{pmatrix} 0 & p_i \\ q_i & 0 \end{pmatrix}$ for $i \geq r+1$, and 0 elsewhere. The corresponding rows and columns are labelled by x_i, y_i, z_i for $1 \leq i \leq r$ and x_i, y_i for $r+1 \leq i \leq k$. Let $F := \det(M + Z)$ and H be its part of total degree at most 2. Then

$$\begin{aligned} H = & \sum_{i=1, (a_i, b_i)}^k (-q_i M_{a_i}^{b_i} - p_i M_{b_i}^{a_i}) \\ & + \sum_{1 \leq i < j \leq k, (a_i, b_i), (a_j, b_j)} (p_i p_j M_{b_i, b_j}^{a_i, a_j} + q_i q_j M_{a_i, a_j}^{b_i, b_j} + p_i q_j M_{b_i, a_j}^{a_i, b_j} + q_i p_j M_{a_i, b_j}^{b_i, a_j}) \\ & + \sum_{i=1}^r p_i^2 M_{x_i, y_i}^{y_i, z_i} + q_i^2 M_{y_i, z_i}^{x_i, y_i}, \end{aligned}$$

where there is an extra summation over (a_i, b_i) for each $i \leq r$ for which $(a_i, b_i) = (x_i, y_i)$ or (y_i, z_i) ; if $i > r$, (a_i, b_i) just means (x_i, y_i) .

Proof. The proof follows by expanding along rows and columns for each rows/columns indexed by x_i, y_i, z_i . \square

3.4. Probability estimates. We won't need the following lemma to compute moments, but state it for its relevance to the quantities computed and its own interest.

Lemma 6. *Let \mathcal{G} be any connected finite graph. The probability that an edge e is in the cycle of $\Gamma_{\mathcal{G}}$ is*

$$\mathbb{P}(e \in \Gamma_{\mathcal{G}}) = \frac{\kappa_{\mathcal{G}}}{\lambda_{\mathcal{G}}} (1 - T_{\mathcal{G}}(e, e)),$$

where $T_{\mathcal{G}}$ is the transfer impedance.

Proof. Consider the bundle Laplacian with parallel transport $q \neq 1$ on edge e (with either orientation), and 1 on all other edges. We have $\det \Delta_e(q) = (2 - q - 1/q)N_e$, where N_e is the number of CRSTs containing e in its cycle. Hence

$$\begin{aligned} \mathbb{P}(e \in \mathbf{\Gamma}_{\mathcal{G}}) &= \frac{N_e}{\lambda_{\mathcal{G}}} = \frac{1}{\lambda_{\mathcal{G}}} \frac{\det \Delta_e(q)}{(2 - q - 1/q)} \\ &= \frac{\kappa_{\mathcal{G}}}{\lambda_{\mathcal{G}}} (1 - T(e, e)), \end{aligned}$$

using Lemma 4 (note that Δ_e differs from the standard Laplacian by a single 2×2 block $\begin{pmatrix} 0 & 1 - q \\ 1 - q^{-1} & 0 \end{pmatrix}$), relation (1), and the matrix-tree theorem. \square

The next two lemmas give the probabilities $\mathbb{P}(f \in \mathbf{\Gamma}_{\mathcal{G}})$ and $\mathbb{P}(f_1, f_2 \in \mathbf{\Gamma}_{\mathcal{G}})$ in terms of the potential theory properties of the graph.

Lemma 7. *Let \mathcal{G} be any connected planar finite graph. Let f be a bounded face. Let γ_f be a simple dual path from f to the unbounded face. Denote by $E(\gamma_f)$ the set of edges the path γ_f crosses, oriented counter-clockwise around f . Then*

$$(3) \quad \mathbb{P}(f \in \mathbf{\Gamma}_{\mathcal{G}}) = \frac{\kappa_{\mathcal{G}}}{\lambda_{\mathcal{G}}} \sum_{a \in E(\gamma_f)} \left(1 - \sum_{b \in E(\gamma_f)} T_{\mathcal{G}}(a, b) \right) = \frac{\kappa_{\mathcal{G}}}{\lambda_{\mathcal{G}}} \sum_{a \in E(\gamma_f), b \in E(\tilde{\gamma}_f)} T_{\mathcal{G}}(a, b),$$

where $\tilde{\gamma}_f$ is a dual path from f to the boundary such that for each edge $e = xy$ in $E(\gamma_f)$, the path $\gamma_f \cup \tilde{\gamma}_f$ disconnects x from y in \mathcal{G} .

Proof. For a face f of \mathcal{G} , and a real or complex parameter $q \neq 0$, define a connection on a discrete line bundle over \mathcal{G} by taking the parallel transport equal to the identity on all edges except those crossed by γ_f ; on these, take parallel transport q (oriented consistently in such a way that the monodromy on any counter-clockwise loop around f is q). See Figure 2. Let $\Delta_f(q)$ be the corresponding bundle Laplacian.

We have

$$\det \Delta_f(q) = \sum_{k=1}^{\infty} N_k (2 - q - 1/q)^k,$$

where N_k is the number of CRSFs with k cycles each of which winds around f . Thus, the probability of the face f being inside the cycle of a uniform CRST is

$$(4) \quad \mathbb{P}(f \in \mathbf{\Gamma}_{\mathcal{G}}) = \frac{N_1}{\lambda_{\mathcal{G}}} = \frac{1}{\lambda_{\mathcal{G}}} \lim_{q \rightarrow 1} \frac{\det \Delta_f(q)}{2 - q - 1/q}.$$

Let $e_i = x_i y_i$ be the elements of the set $E(\gamma_f)$ and $k = |E(\gamma_f)|$ its cardinality. Let Δ be the matrix of the combinatorial Laplacian where we have chosen to order the vertices of \mathcal{G} starting with $x_1, y_1, x_2, y_2, \dots, x_k, y_k$. Let p_i, q_i be $2k$ variables and let Z

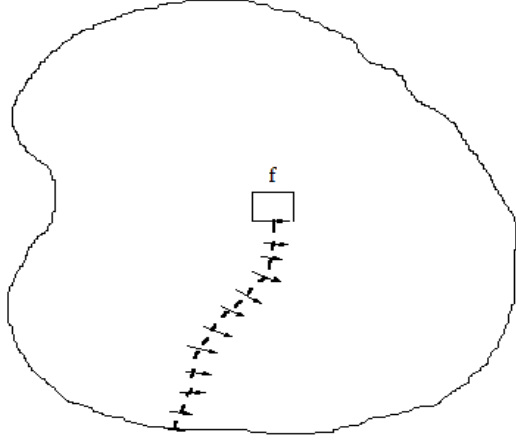


FIGURE 2. Zipper connection with parameter q for a fixed face f .

be the block-diagonal matrix, whose first k blocks are $\begin{pmatrix} 0 & q_i \\ p_i & 0 \end{pmatrix}$ and zero elsewhere. Now set F to be the determinant of $\Delta + Z$. This is a polynomial in the variables p_i, q_i . Let H be its degree-at-most-two part as in Lemma 4.

By Lemma 4,

$$H = \sum_{i=1}^k (-q_i \Delta_{x_i}^{y_i} - p_i \Delta_{y_i}^{x_i} - p_i q_i \Delta_{x_i, y_i}^{x_i, y_i}) + \sum_{1 \leq i < j \leq k} (p_i p_j \Delta_{y_i, y_j}^{x_i, x_j} + q_i q_j \Delta_{x_i, x_j}^{y_i, y_j} + p_i q_j \Delta_{y_i, x_j}^{x_i, y_j} + q_i p_j \Delta_{x_i, y_j}^{y_i, x_j}).$$

Using the symmetry of the matrix Δ , this yields

$$H = \sum_{i=1}^k (-(p_i + q_i) \Delta_{x_i}^{y_i} - p_i q_i \Delta_{x_i, y_i}^{x_i, y_i}) + \sum_{1 \leq i < j \leq k} ((p_i p_j + q_i q_j) \Delta_{x_i, x_j}^{y_i, y_j} + (p_i q_j + q_i p_j) \Delta_{x_i, y_j}^{y_i, x_j}).$$

Let $\varepsilon > 0$ be small, and set $p_i = 1 - q = -\varepsilon$, $q_i = 1 - 1/q = \varepsilon - \varepsilon^2 + O(\varepsilon^3)$. We now view F and H as functions of ε .

Recalling Theorem 1, $F(\varepsilon) = H(\varepsilon) + O(\varepsilon^3)$, where

$$H(\varepsilon)/(-\varepsilon^2) = k\kappa_{\mathcal{G}} - \sum_{i=1}^k \Delta_{x_i, y_i}^{x_i, y_i} - 2 \sum_{1 \leq i < j \leq k} (-\Delta_{x_i, x_j}^{y_i, y_j} + \Delta_{x_i, y_j}^{y_i, x_j}) + O(\varepsilon).$$

Using Lemma 1 and the definition of T ,

$$\Delta_{y_i, x_j}^{x_i, y_j} - \Delta_{y_i, y_j}^{x_i, x_j} = \kappa_{\mathcal{G}}(G(x_i, y_j) + G(x_j, y_i) - G(x_i, x_j) - G(y_i, y_j)) = -\kappa_{\mathcal{G}}T(e_i, e_j),$$

and

$$\Delta_{x_i, y_i}^{x_i, y_i} = \kappa_{\mathcal{G}}(G(x_i, x_i) + G(y_i, y_i) - G(x_i, y_i) - G(x_i, y_i)) = \kappa_{\mathcal{G}}T(e_i, e_i).$$

We obtain that

$$\det \Delta_f(1 - \varepsilon)/(-\varepsilon^2) = \kappa_{\mathcal{G}} \sum_{i=1}^k \left(1 - \sum_{j=1}^k T(e_i, e_j)\right) + O(\varepsilon).$$

The first equality in (3) now follows from (4). The second equality follows from Lemma 2. \square

Lemma 8. *Let \mathcal{G} be any connected planar finite graph. Consider two bounded faces f and f' and two non-intersecting simple dual paths γ_f (resp. $\gamma_{f'}$) from f to the unbounded face (resp. from f' to the unbounded face). Let $E(\gamma_f), E(\gamma_{f'})$ be the sets of edges crossed by γ_f and $\gamma_{f'}$, oriented counter-clockwise around f, f' respectively. Then*

$$\mathbb{P}(f, f' \in \Gamma_{\mathcal{G}}) = -\frac{\kappa_{\mathcal{G}}}{\lambda_{\mathcal{G}}} \sum_{a \in E(\gamma_f), b \in E(\gamma_{f'})} T_{\mathcal{G}}(a, b).$$

Proof. Take a connection with monodromy q around f and q' around f' as before, supported on $E(\gamma_f)$ and $E(\gamma_{f'})$. See Figure 3.

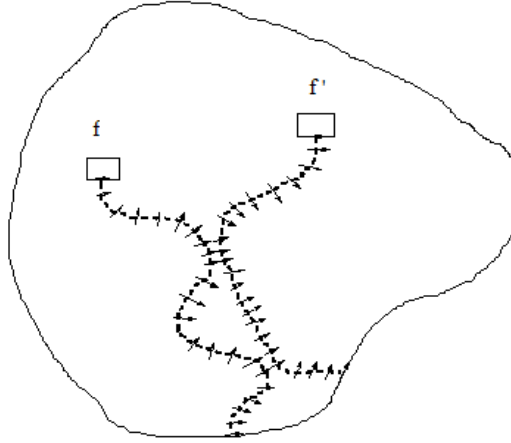


FIGURE 3. Zipper connection with parameters q and q' for two fixed faces f and f' , where the paths potentially have common adjacent vertices.

We have

$$\det \Delta_{f,f'}(q, q') = \sum_{a,b,c} N_{a,b,c} (2 - q - 1/q)^a (2 - q' - 1/q')^b (2 - qq' - 1/(qq'))^c,$$

where $N_{a,b,c}$ is the number of CRSFs of \mathcal{G}_n with $a + b + c$ cycles, a of which contain f but not f' , b of which contain f' but not f , and c of which contain both f and f' . Therefore, $\mathbb{P}(f, f' \in \Gamma_n) = N_{0,0,1}/\lambda_{\mathcal{G}}$.

When $q \rightarrow 1$ we have

$$\det \Delta_{f,f'}(q, q) = (N_{1,0,0} + N_{0,1,0})(2 - q - 1/q) + N_{0,0,1}(2 - q^2 - 1/q^2) + O((q - 1)^2).$$

As $q \rightarrow 1$, we get

$$\lim_{q \rightarrow 1} \frac{\det \Delta_{f,f'}(q, q)}{2 - q - 1/q} = N_{1,0,0} + N_{0,1,0} + 4N_{0,0,1}.$$

Using the previous section, we also get that

$$\lim_{q \rightarrow 1} \frac{\det \Delta_{f,f'}(q, 1)}{2 - q - 1/q} = N_{1,0,0} + N_{0,0,1},$$

and

$$\lim_{q \rightarrow 1} \frac{\det \Delta_{f,f'}(1, q)}{2 - q - 1/q} = N_{0,1,0} + N_{0,0,1}.$$

Hence

$$N_{0,0,1} = \frac{1}{2} \left(\lim_{q \rightarrow 1} \frac{\det \Delta_{f,f'}(q, q)}{2 - q - 1/q} - \lim_{q \rightarrow 1} \frac{\det \Delta_{f,f'}(q, 1)}{2 - q - 1/q} - \lim_{q \rightarrow 1} \frac{\det \Delta_{f,f'}(1, q)}{2 - q - 1/q} \right).$$

Let us first deal with the case where the paths γ_f and $\gamma_{f'}$ don't have any common adjacent vertex. In that case, we deduce from Lemma 7 that the contribution is $-\kappa_{\mathcal{G}}/\lambda_{\mathcal{G}} \sum_{i,j} T(e_i, e'_j)$, where the sum is over all e_i and e'_j in $E(\gamma_f)$ and $E(\gamma_{f'})$, respectively.

Let us now consider the case where the paths γ_f and $\gamma_{f'}$ are adjacent to common vertices, say r of them. We label these r 2-edges-paths by ordered triplets of vertices x_i, y_i, z_i for $i \leq r$. The rest of the edges of $E(\gamma_f)$ are labelled $x_i y_i$ for $i \in I$ and the rest of the edges of $E(\gamma_{f'})$ are labelled x'_j, y'_j for $j \in I'$.

The Laplacian with parallel transport q on $E(\gamma_f)$ and q' on $E(\gamma_{f'})$ can be written as $\Delta + Z$ where Z is the following sparse block-diagonal matrix. The non-zero blocks

of Z consist in r blocks of the form $\begin{pmatrix} 0 & p_i & 0 \\ q_i & 0 & p_i \\ 0 & q_i & 0 \end{pmatrix}$ indexed by rows/columns x_i, y_i, z_i

for $i \leq r$ and $k - r$ blocks of the form $\begin{pmatrix} 0 & p_i \\ q_i & 0 \end{pmatrix}$ for $r + 1 \leq i \leq k$, where we ultimately replace p_i by $1 - q = -\varepsilon$ and q_i by $1 - 1/q = \varepsilon + O(\varepsilon^2)$.

Using Lemma 5, we can compute the degree at most 2 part of the differences of four determinants of Laplacians $\det(\Delta + Z)$ seen as polynomials in the variables p_i, q_i . After cancellations, the remaining terms of degree at most 2 are for p_i^2, q_i^2 for $i \leq r$; $p_i p_j, q_i q_j, p_i q_j$ and $q_i p_j$, for $i < j$ and appropriate pairings.

We replace $p_i = 1 - q = -\varepsilon$ and $q_i = 1 - 1/q = \varepsilon + O(\varepsilon^2)$. The total contribution (dividing by $2 - q - 1/q = -\varepsilon^2$) is

$$2 \sum_{i \in E(\gamma_f), j \in E(\gamma_{f'})} \Delta_{x_i, y'_j}^{y_i, x'_j} - \Delta_{x_i, x'_j}^{y_i, y'_j} = -2\kappa_{\mathcal{G}} \sum_{i \in E(\gamma_f), j \in E(\gamma_{f'})} T(e_i, e'_j)$$

where the right hand side was obtained using Lemma 1 and the definition of T . Dividing by 2 we get the result. \square

Remark 1. *It is worth noting that the right-hand sides of Lemma 7 and Lemma 8 do not depend on the particular choice of paths γ_f and $\gamma_{f'}$.*

4. ESTIMATES FOR FINITE SQUARE REGIONS IN \mathbb{Z}^2

Let \mathcal{G}_n be the $n \times n$ square grid graph. Its vertices are indexed by $[1, n] \times [1, n]$ and edges join nearest neighbors. We denote by κ_n and λ_n the number of spanning trees and cycle-rooted trees of \mathcal{G}_n , and by $\mathbf{\Gamma}_n$, \mathbf{A}_n and \mathbf{L}_n the uniform CRST on \mathcal{G}_n , its area and its length.

4.1. Explicit expression for the transfer impedance. Let V_0 denote the space of zero-mean real-valued functions on the vertices of \mathcal{G}_n .

The family of functions

$$f_{k,l}(x, y) = \frac{2}{n(1 + \mathbf{1}_{k=0} + \mathbf{1}_{l=0})} \cos(\pi k(x - 1/2)/n) \cos(\pi l(y - 1/2)/n)$$

for $(k, l) \in \{0, \dots, n-1\}^2 \setminus \{(0, 0)\}$ forms an orthonormal basis of V_0 which consists of eigenvectors for $\Delta_{|V_0}$ corresponding to the eigenvalues $\lambda_{k,l} = 4(\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})) \neq 0$.

Hence, using (2), we obtain that for any two horizontal edges $e_i = ((x, y_i), (x + 1, y_i))$ and $e'_j = ((x', y_j), (x' + 1, y_j))$,

$$\begin{aligned} T(e_i, e'_j) &= \frac{4}{n^2} \sum_{k,l=1}^{n-1} \sin\left(\frac{\pi k x}{n}\right) \sin\left(\frac{\pi k x'}{n}\right) \frac{\sin^2\left(\frac{\pi k}{2n}\right) \cos\left(\frac{\pi l(y_i - 1/2)}{n}\right) \cos\left(\frac{\pi l(y_j - 1/2)}{n}\right)}{\sin^2(\pi k/(2n)) + \sin^2(\pi l/(2n))} \\ (5) \quad &+ \frac{2}{n^2} \sum_{k=1}^{n-1} \sin\left(\frac{\pi k x}{n}\right) \sin\left(\frac{\pi k x'}{n}\right). \end{aligned}$$

In particular, if $x = x'$, we have

$$(6) \quad T(e_i, e'_j) = \frac{1}{n} + \frac{4}{n^2} \sum_{k,l=1}^{n-1} \sin^2\left(\frac{\pi k x}{n}\right) \frac{\sin^2\left(\frac{\pi k}{2n}\right) \cos\left(\frac{\pi l(y_i-1/2)}{n}\right) \cos\left(\frac{\pi l(y_j-1/2)}{n}\right)}{\sin^2(\pi k/(2n)) + \sin^2(\pi l/(2n))},$$

since for any $x \neq 0 \pmod n$, we have $\sum_{k=1}^{n-1} \sin^2\left(\frac{\pi k x}{n}\right) = \frac{n}{2}$.

4.2. Expected area. Recall that the expected area of Γ_n is

$$\mathbb{E}(\mathbf{A}_n) = \sum_f \mathbb{P}(f \in \Gamma_n),$$

where the sum is over all $(n-1)^2$ bounded faces of \mathcal{G}_n .

The probabilities that appear in the sum may be computed with the help of the following lemma which is a special case of Lemma 7.

Lemma 9. *Let f be a bounded face of \mathcal{G}_n and denote by k the y -coordinate of its lower vertices. Denote by $e_i = a_i b_i$ the k east-oriented edges below f , enumerated from bottom to top. Then*

$$\mathbb{P}(f \in \Gamma_n) = \frac{\kappa_n}{\lambda_n} \left(k - \sum_{i,j=1}^k T(e_i, e_j) \right),$$

Equivalently,

$$\mathbb{P}(f \in \Gamma_n) = \frac{\kappa_n}{\lambda_n} \sum_{i=1}^k u_f(i),$$

where $u_f(i)$ is the amount of current that flows left of f when \mathcal{G}_n is considered as a resistor network with all edges having resistance one, and when one unit of current flows into a_i and out of b_i .

Proof. We consider a vertical zipper of edges below face f , see Figure 4, and use Lemma 7. \square

Remark 2. *It is interesting to note that if f' denotes the face with the same x -coordinate as f but with y -coordinate equal to $n-k$, the probabilities $\mathbb{P}(f \in \Gamma_n)$ and $\mathbb{P}(f' \in \Gamma_n)$ being equal for symmetry reasons, we obtain that*

$$\sum_{i=1}^k u_f(i) = \sum_{i=1}^{n-k} u_{f'}(i).$$

Theorem 3. *As $n \rightarrow \infty$, we have*

$$\mathbb{E}(\mathbf{A}_n) = 4/\pi \log n + o(\log n).$$

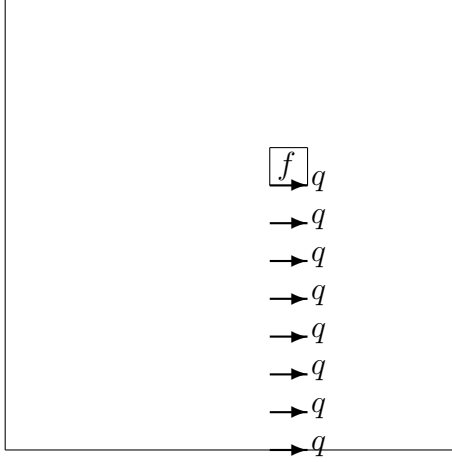


FIGURE 4. Zipper connection with parameter q for a fixed face f .

Proof. Recall that $\frac{\kappa_n}{\lambda_n} = \frac{8}{n^2}(1+o(1))$. Therefore (with $a_n \sim b_n$ denoting $\lim a_n/b_n = 1$)

$$\begin{aligned}
\mathbb{E}(\mathbf{A}_n) &= \sum_f \mathbb{P}(f \in \Gamma_n) \\
&= \frac{\kappa_n}{\lambda_n} \sum_{x,y=1}^{n-1} \left(y - \sum_{y_i=1}^y \sum_{y_j=1}^y T(e_i, e_j) \right) \\
&= \frac{\kappa_n}{\lambda_n} \left(\frac{n(n-1)^2}{2} - \sum_{x,y=1}^{n-1} \sum_{y_i=1}^y \sum_{y_j=1}^y T(e_i, e_j) \right) \\
&= \frac{8}{n^2}(1+o(1)) \left(\frac{n(n-1)^2}{2} - \left(\frac{n^3}{2} - \frac{n^2}{2\pi} \log n + O(n^2) \right) \right) \\
&= \frac{4 \log n}{\pi} + o(\log n),
\end{aligned}$$

where we have simplified (using (6) and some trigonometric identities)

$$\sum_{x,y=1}^{n-1} \sum_{y_i=1}^y \sum_{y_j=1}^y T(e_i, e_j) =$$

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$$\begin{aligned}
&= \sum_{x,y=1}^{n-1} \sum_{y_i=1}^y \sum_{y_j=1}^y \left(1/n + \frac{4}{n^2} \sum_{k,l \neq 0} \sin^2\left(\frac{\pi k}{2n}\right) \sin^2\left(\frac{\pi kx}{n}\right) \frac{\cos\left(\frac{\pi l(y_i - \frac{1}{2})}{n}\right) \cos\left(\frac{\pi l(y_j - \frac{1}{2})}{n}\right)}{\sin^2(\pi k/(2n)) + \sin^2(\pi l/(2n))} \right) \\
&= \frac{(n-1)^2(2n-1)}{6} + \frac{4}{n^2} \sum_{k=1}^{n-1} \left(\sum_{x=1}^{n-1} \sin^2\left(\frac{\pi kx}{n}\right) \right) \left(\sum_{l=1}^{n-1} \frac{\sin^2\left(\frac{\pi k}{2n}\right)}{\sin^2(\pi k/(2n)) + \sin^2(\pi l/(2n))} \right) \\
&\quad \times \sum_{y=1}^{n-1} \sum_{y_i=1}^y \cos\left(\frac{\pi l(y_i - \frac{1}{2})}{n}\right) \sum_{y_j=1}^y \cos\left(\frac{\pi l(y_j - \frac{1}{2})}{n}\right) \\
&= \frac{(n-1)^2(2n-1)}{6} + \frac{4}{n^2} \sum_{k=1}^{n-1} \frac{n}{2} \left(\sum_{l=1}^{n-1} \frac{\sin^2\left(\frac{\pi k}{2n}\right)}{\sin^2(\pi k/(2n)) + \sin^2(\pi l/(2n))} \left(\frac{n}{8 \sin^2(\frac{l\pi}{2n})} \right) \right) \\
&= \frac{(n-1)^2(2n-1)}{6} + \frac{1}{4} \sum_{k,l=1}^{n-1} \frac{\sin^2\left(\frac{\pi k}{2n}\right) / \sin^2\left(\frac{\pi l}{2n}\right)}{\sin^2(\pi k/(2n)) + \sin^2(\pi l/(2n))} \\
&= \frac{n^3}{2} - \frac{n^2}{2\pi} \log n + O(n^2).
\end{aligned}$$

The last line can be seen as follows. Using that $\frac{a/b}{a+b} + \frac{b/a}{a+b} = \frac{1}{a} + \frac{1}{b} - \frac{2}{a+b}$ and symmetrizing the sum in k, l we obtain that

$$\begin{aligned}
\sum_{k,l=1}^{n-1} \frac{\sin^2\left(\frac{\pi k}{2n}\right) / \sin^2\left(\frac{\pi l}{2n}\right)}{\sin^2(\pi k/(2n)) + \sin^2(\pi l/(2n))} &= (n-1) \sum_{k=1}^{n-1} \frac{1}{\sin^2\left(\frac{\pi k}{2n}\right)} - \sum_{k=1}^{n-1} \sum_{l=1}^{n-1} \frac{1}{\sin^2\left(\frac{\pi k}{2n}\right) + \sin^2\left(\frac{\pi l}{2n}\right)} \\
&= \frac{2}{3} n^3 - \frac{2n^2}{\pi} \log n + O(n^2),
\end{aligned}$$

where the asymptotics follow from Lemmas 10 and 11 below. \square

Lemma 10.

$$\sum_{k=1}^{n-1} \sum_{l=1}^{n-1} \frac{1}{\sin^2(\pi k/(2n)) + \sin^2(\pi l/(2n))} = \frac{2n^2}{\pi} \log n + O(n^2).$$

Proof. We consider $\varepsilon \in (0, 1)$. We may rewrite the sum as one fourth of the sum over pairs of integers $(k, l) \neq (0, 0)$ where k and l run from $-(n-1)$ to $n-1$. Now we separate the sum in two contributions.

One for which $k^2 + l^2$ is greater than εn^2 . This gives a total contribution of $O(n^2/\varepsilon)$. On the other part, where $k^2 + l^2$ is smaller than εn^2 , we may use a series expansion. We have

$$\sin^2\left(\frac{\pi k}{2n}\right) + \sin^2\left(\frac{\pi l}{2n}\right) = \frac{\pi^2}{4n^2} (k^2 + l^2) \left(1 + O\left(\frac{k^2 + l^2}{n^2}\right) \right) = \frac{\pi^2}{4n^2} (k^2 + l^2) (1 + O(\varepsilon)).$$

Hence,

$$\sum_{k=1}^{n-1} \sum_{l=1}^{n-1} \frac{1}{\sin^2(\pi k/n) + \sin^2(\pi l/n)} = \frac{n^2}{\pi^2} \sum_{k^2+l^2 \leq \varepsilon n^2} \frac{1}{k^2+l^2} (1 + O(\varepsilon^2)) + O(n^2/\varepsilon).$$

Observe the simple geometrical fact that the cardinality of the set $D(r) = \{(k, l) \in \mathbb{Z}^2, k^2 + l^2 \leq r\}$ satisfies $\pi(\sqrt{n} - \sqrt{2})^2 \leq D(r) \leq \pi(\sqrt{n} + \sqrt{2})^2$. Hence $D(r+1) - D(r) = \pi + O(1/\sqrt{r})$. Furthermore, if $r-1 \leq k^2 + l^2 \leq r$, we have $1/(k^2 + l^2) = 1/r(1 + O(1/r))$. Therefore,

$$\begin{aligned} \sum_{k^2+l^2 \leq \varepsilon n^2} \frac{1}{k^2+l^2} &= \sum_{r=1}^{\varepsilon n^2} \frac{\pi}{r} (1 + O(\frac{1}{\sqrt{r}})) \\ &= \pi \log(\varepsilon n^2) + O(1) \\ &= 2\pi \log n + O(1 + \log \varepsilon), \end{aligned}$$

Hence,

$$\sum_{k=1}^{n-1} \sum_{l=1}^{n-1} \frac{1}{\sin^2(\pi k/(2n)) + \sin^2(\pi l/(2n))} = \frac{2n^2}{\pi} \log n + O(n^2/\varepsilon^2).$$

□

Lemma 11.

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2(\frac{\pi k}{2n})} = \frac{2}{3}(n^2 - 1).$$

Proof. Let $p(z) = \prod_{k=1}^{n-1} (z - 4 \sin^2(\frac{\pi k}{2n}))$. The desired quantity is $-4 \frac{p'(0)}{p(0)}$. Writing $p(z) = \prod_{k=1}^{n-1} (z - 2(1 - \cos \frac{\pi k}{n}))$ we see that $p(2z+2)$ has roots $\cos(\pi k/n)$, that is, it is the Chebyshev polynomial of the second kind $\text{Ch}_{n-1}(z)$, which can also be defined by

$$(7) \quad \text{Ch}_{n-1}(\cos \theta) = \frac{\sin n\theta}{\sin \theta}.$$

We then have

$$-4 \frac{p'(0)}{p(0)} = -2 \frac{\text{Ch}'_{n-1}(-1)}{\text{Ch}_{n-1}(-1)}$$

and the result follows from differentiating (7). □

4.3. **Second moment of the area.** Recall that

$$\mathbb{E}(\mathbf{A}_n^2) = \sum_{f, f'} \mathbb{P}(f, f' \in \mathbf{\Gamma}_n),$$

where the sum is over all bounded faces of \mathcal{G}_n .

These probabilities can be computed by Lemma 8 using the vertical zippers illustrated in Figure 5. Denote by e_i the k east-oriented edges below f enumerated from bottom to top. Similarly, denote by e'_i the k' edges below f' . When f and f' are not in the same column, we have

$$\mathbb{P}(f, f' \in \mathbf{\Gamma}_n) = -\frac{\kappa_n}{\lambda_n} \sum_{i=1}^k \sum_{j=1}^{k'} T(e_i, e'_j).$$

When they are in the same column we rotate the square, interchanging columns and rows.

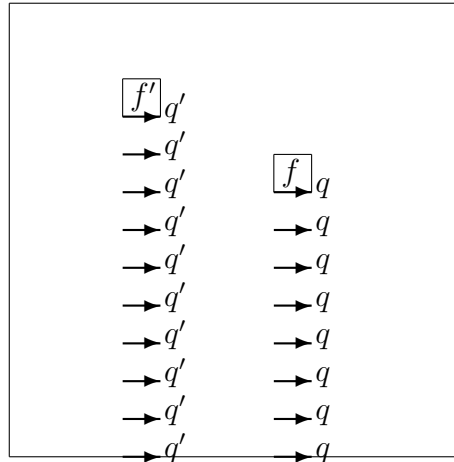


FIGURE 5. Zipper connection with parameters q and q' for two fixed faces f and f' .

Theorem 4. As $n \rightarrow \infty$,

$$\mathbb{E}(\mathbf{A}_n^2) = Cn^2 + o(n^2),$$

where $C = \frac{512\beta}{\pi^6} \approx .281$, where

$$\beta = \sum_{k, l=1, \text{ both odd}}^{\infty} \frac{1}{k^2 l^2 (k^2 + l^2)} \approx .527.$$

Proof. By symmetry, the sum $\sum_{f,f'} \mathbb{P}(f, f' \in \Gamma)$ over all ordered pairs of faces f and f' may be rewritten as the sum of three contributions: a sum over f, f' lying in different rows and columns; twice the sum over faces lying in same rows and different columns; and a sum over faces $f = f'$.

Therefore, using the parametrization of faces f by the coordinates (x, y) of their lower left vertex, we may write

(8)

$$\mathbb{E}(\mathbf{A}_n^2) = 4 \sum_{x,y=1}^{n-1} \sum_{x'>x, y'>y} \mathbb{P}(f, f' \in \Gamma_n) + 4 \sum_{x,y=1}^{n-1} \sum_{y=y', x'>x} \mathbb{P}(f, f' \in \Gamma_n) + \sum_{y=y', x=x'} \mathbb{P}(f \in \Gamma_n).$$

Let us compute the first sum in (8). Define

$$I_1 = \frac{\lambda_{\mathcal{G}}}{\kappa_{\mathcal{G}}} \sum_{x,y=1}^{n-1} \sum_{x'>x, y'>y} \mathbb{P}(f, f' \in \Gamma_n) = - \sum_{x,y=1}^{n-1} \sum_{x'>x, y'>y} \sum_{y_i=1}^y \sum_{y'_j=1}^{y'} T(e_i, e'_j).$$

Recall the expression (5) for $T(e_i, e'_j)$. It follows that I_1 only depends on x, x' through the term

$$A(n, k) := \sum_{\substack{x, x'=1, \\ x'>x}}^{n-1} \sin\left(\frac{\pi k x}{n}\right) \sin\left(\frac{\pi k x'}{n}\right) = \frac{-n + (1 - (-1)^k) \cot^2(\frac{\pi k}{2n})}{4}$$

by Lemma 12. So

$$\begin{aligned} I_1 = & - \sum_{y=1}^{n-1} \sum_{y'>y} \frac{2yy'}{n^2} \sum_{k=1}^{n-1} A(n, k) - \\ & - \frac{4}{n^2} \sum_{\substack{y, y'=1, \\ y'>y}}^{n-1} \sum_{k, l=1}^{n-1} A(n, k) \left(\frac{\sin^2(\frac{\pi k}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \sum_{y_i=1}^y \cos\left(\frac{\pi l(y_i - 1/2)}{n}\right) \sum_{y'_j=1}^{y'} \cos\left(\frac{\pi l(y'_j - 1/2)}{n}\right) \right) \end{aligned}$$

and using a simple trigonometric identity in the second line

$$\begin{aligned}
&= -\frac{(3n^2 - 7n + 2)n(n-1)}{12n^2} \sum_{k=1}^{n-1} A(n, k) \\
&\quad - \frac{4}{n^2} \sum_{\substack{y, y'=1 \\ y' > y}}^{n-1} \sum_{k, l=1}^{n-1} A(n, k) \left(\frac{\sin^2(\frac{\pi k}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \frac{\sin(l\pi y/n) \sin(l\pi y'/n)}{4 \sin^2(\frac{l\pi}{2n})} \right) \\
&= \frac{(3n^2 - 7n + 2)n(n-1)}{48n^2} \left(n(n-1) - 2 \sum_{k=1, k \text{ odd}}^{n-1} \cot^2(\frac{\pi k}{2n}) \right) - \\
&\quad - \frac{4}{n^2} \sum_{k, l=1}^{n-1} A(n, k) \left(\frac{\sin^2(\frac{\pi k}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \frac{A(n, l)}{4 \sin^2(\frac{l\pi}{2n})} \right) \\
&= -\frac{4}{n^2} \sum_{k, l=1}^{n-1} A(n, k) \left(\frac{\sin^2(\frac{\pi k}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \frac{A(n, l)}{4 \sin^2(\frac{l\pi}{2n})} \right) + O(n^3),
\end{aligned}$$

where we have used Lemma 16 in the last line.

Now expanding the $A(n, \cdot)$ yields

$$\begin{aligned}
I_1 &= -\frac{1}{16n^2} \left(n^2 \sum_{k, l=1}^{n-1} \frac{\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} - 2n \sum_{k, l=1, l \text{ odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n}) \right) \cot^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \right. \\
&\quad - 2n \sum_{k, l=1, k \text{ odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n}) \right) \cot^2(\frac{\pi k}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \\
&\quad \left. + 4 \sum_{k, l=1, k \text{ odd}, l \text{ odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n}) \right) \cot^2(\frac{\pi k}{2n}) \cot^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \right) + O(n^3) \\
&= \frac{1}{8n} \sum_{k, l=1, l \text{ odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n}) \right) \cot^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} + \frac{1}{8n} \sum_{k, l=1, k \text{ odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n}) \right) \cot^2(\frac{\pi k}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \\
&\quad - \frac{1}{4n^2} \sum_{k, l=1, k \text{ odd}, l \text{ odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n}) \right) \cot^2(\frac{\pi k}{2n}) \cot^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} + O(n^3),
\end{aligned}$$

where we used that $\sum_{k,l=1}^{n-1} \frac{\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n})+\sin^2(\frac{\pi l}{2n})} = O(n^3)$, as shown in the proof of Theorem 3.

Now, using Lemmas 13, 17, and 18, we obtain that

$$I_1 = \frac{1}{8n}(n^5/6 + O(n^4)) - \frac{1}{4n^2}(1/12 - 2^6\beta/\pi^6)(n^6 + O(n^5)) + O(n^3) = \frac{16\beta}{\pi^6}n^4 + O(n^3),$$

and hence

$$4 \sum_{x,y=1}^{n-1} \sum_{x'>x, y'>y} \mathbb{P}(f, f' \in \Gamma_n) = \frac{4\kappa_{\mathcal{G}}}{\lambda_{\mathcal{G}}} I_1 = \frac{32}{n^2}(1+o(1))\left(\frac{16\beta}{\pi^6}n^4 + O(n^3)\right) = \frac{512\beta}{\pi^6}n^2 + o(n^2).$$

Let us now compute the second sum of (8). Define

$$I_2 = \frac{\lambda_{\mathcal{G}}}{\kappa_{\mathcal{G}}} \sum_{x,y=1}^{n-1} \sum_{x'>x, y'=y} \mathbb{P}(f, f' \in \Gamma_n) = - \sum_{x,y=1}^{n-1} \sum_{x'>x, y'=y} \sum_{y_i=1}^y \sum_{y'_j=1}^y T(e_i, e'_j)$$

We have

$$\begin{aligned} I_2 &= - \sum_{x,y=1}^{n-1} \sum_{x'>x} \frac{2y^2}{n^2} \sum_{k=1}^{n-1} \sin\left(\frac{\pi kx}{n}\right) \sin\left(\frac{\pi kx'}{n}\right) - \frac{4}{n^2} \sum_{y=1}^{n-1} \sum_{k,l=1}^{n-1} \left(\sum_{\substack{x,x'=1 \\ x'>x}}^{n-1} \sin\left(\frac{\pi kx}{n}\right) \sin\left(\frac{\pi kx'}{n}\right) \right) \\ &\quad \left(\frac{\sin^2(\frac{\pi k}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \sum_{y_i=1}^y \cos\left(\frac{\pi l(y_i - 1/2)}{n}\right) \sum_{y_j=1}^y \cos\left(\frac{\pi l(y_j - 1/2)}{n}\right) \right) \\ &= - \frac{2n(n-1)(2n-1)}{6n^2} \sum_{k=1}^{n-1} A(n, k) - \frac{4}{n^2} \sum_{y=1}^{n-1} \sum_{k,l=1}^{n-1} A(n, k) \left(\frac{\sin^2(\frac{\pi k}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \frac{\sin^2(l\pi y/n)}{4 \sin^2(\frac{l\pi}{2n})} \right) \\ &= - \frac{4}{n^2} \sum_{k,l=1}^{n-1} A(n, k) \left(\frac{\sin^2(\frac{\pi k}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \frac{n}{8 \sin^2(\frac{l\pi}{2n})} \right) + O(n^3) \\ &= \frac{1}{8} \sum_{k,l=1}^{n-1} \frac{\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} - \frac{1}{4n} \sum_{k,l=1, k \text{ odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n}) \right) \cot^2(\frac{k\pi}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} + O(n^3) \\ &= - \frac{1}{4n} \sum_{k,l=1, k \text{ odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n}) \right) \cot^2(\frac{k\pi}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} + O(n^3), \end{aligned}$$

where we again used that $\sum_{k,l=1}^{n-1} \frac{\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n})+\sin^2(\frac{\pi l}{2n})} = O(n^3)$. Using Lemma 17, we thus have $I_2 = O(n^3)$; multiplying by $\frac{\kappa_{\mathcal{G}}}{\lambda_{\mathcal{G}}}$ we get that

$$\sum_{x,y=1}^{n-1} \sum_{x'>x, y'=y} \mathbb{P}(f, f' \in \Gamma_n) = O(n).$$

The third term of (8) is $O(\log n)$ by Theorem 3. Therefore the only term of (8) that contributes to the first-order asymptotics of the sum is the first one, and we get the result. \square

Lemma 12.

$$\sum_{\substack{x,x'=1 \\ x'>x}}^{n-1} \sin\left(\frac{\pi kx}{n}\right) \sin\left(\frac{\pi kx'}{n}\right) = \frac{-n + (1 - (-1)^k) \cot^2(\frac{\pi k}{2n})}{4}.$$

Proof. This is obtained through expansion in terms of exponentials. \square

Lemma 13.

$$\sum_{k,l=1, \text{ both odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n})\right) \cot^2(\frac{\pi k}{2n}) \cot^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} = \left(\frac{1}{12} - \frac{2^6 \beta}{\pi^6}\right) n^6 + O(n^5),$$

where

$$\beta = \sum_{k,l=1, \text{ both odd}}^{\infty} \frac{1}{k^2 l^2 (k^2 + l^2)}.$$

Proof. Using that

$$\frac{(a^2/b^2)(c^2/a^2)(d^2/b^2)}{a^2 + b^2} + \frac{(b^2/a^2)(c^2/a^2)(d^2/b^2)}{a^2 + b^2} = \frac{c^2 d^2 (a^4 + b^4)}{a^4 b^4 (a^2 + b^2)} = \frac{c^2 d^2}{a^2 b^4} + \frac{c^2 d^2}{a^4 b^2} - 2 \frac{c^2 d^2}{a^2 b^2 (a^2 + b^2)},$$

and symmetrizing the sum in k, l , we obtain that

$$\begin{aligned} & \sum_{k,l=1, \text{ both odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n})/\sin^2(\frac{\pi l}{2n})\right) \cot^2(\frac{\pi k}{2n}) \cot^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \\ &= \sum_{k=1, k \text{ odd}}^{n-1} \cot^2\left(\frac{\pi k}{2n}\right) \sum_{k=1, k \text{ odd}}^{n-1} \frac{\cos^2(\frac{\pi k}{2n})}{\sin^4(\frac{\pi k}{2n})} - \sum_{k,l=1, \text{ both odd}}^{n-1} \frac{\cot^2(\frac{\pi k}{2n}) \cot^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} \end{aligned}$$

and using Lemma 16 that

$$\begin{aligned}
&= \frac{n^6}{12} - \sum_{k,l=1, \text{ both odd}}^{n-1} \frac{1}{\sin^2(\frac{\pi k}{2n}) \sin^2(\frac{\pi l}{2n}) (\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n}))} \\
&+ \sum_{k,l=1, \text{ both odd}}^{n-1} \frac{1}{\sin^2(\frac{\pi k}{2n}) \sin^2(\frac{\pi l}{2n})} - \sum_{k,l=1, \text{ both odd}}^{n-1} \frac{1}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} + O(n^5) \\
&= \left(\frac{1}{12} - \frac{2^6 \beta}{\pi^6} \right) n^6 + O(n^5),
\end{aligned}$$

where we have used Lemmas 14 and 15 and the following consequence of Lemma 11

$$\begin{aligned}
\sum_{k,l=1, \text{ both odd}}^{n-1} \frac{1}{\sin^2(\frac{\pi k}{2n}) \sin^2(\frac{\pi l}{2n})} &= \left(\sum_{k=1, k \text{ odd}}^{n-1} \frac{1}{\sin^2(\frac{\pi k}{2n})} \right)^2 \\
&= \left(\frac{4n^2}{\pi^2} \sum_{k \geq 1, k \text{ odd}} \frac{1}{k^2} + O(n) \right)^2 \\
&= \left(\frac{4n^2}{\pi^2} \frac{\pi^2}{6} \left(1 - \frac{1}{4}\right) + O(n) \right)^2 \\
&= \frac{n^4}{4} + O(n^3).
\end{aligned}$$

□

Lemma 14.

$$\sum_{k,l=1, \text{ both odd}}^{n-1} \frac{1}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} = O(n^2 \log n),$$

Proof. This is a consequence of Lemma 10. □

Lemma 15.

$$\sum_{k,l=1, \text{ both odd}}^{n-1} \frac{1}{\sin^2(\frac{\pi k}{2n}) \sin^2(\frac{\pi l}{2n}) (\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n}))} = \frac{2^6 n^6}{\pi^6} \sum_{k,l=1, \text{ both odd}}^{\infty} \frac{1}{k^2 l^2 (k^2 + l^2)} + O(n^5).$$

Proof. The proof follows the same scheme as the beginning of the proof of Lemma 10. □

Lemma 16.

$$\sum_{k=1, k \text{ odd}}^{n-1} \cot^2\left(\frac{\pi k}{2n}\right) = \frac{n^2}{2} + O(n),$$

and

$$\sum_{k=1, k \text{ odd}}^{n-1} \frac{\cos^2(\frac{\pi k}{2n})}{\sin^4(\frac{\pi k}{2n})} = \frac{n^4}{6} + O(n^3).$$

Proof. The proof follows the same scheme as the beginning of the proof of Lemma 10. For the first equality, we obtain that the sum is equivalent to $\frac{4n^2}{\pi^2} \zeta^o(2) = n^2 \frac{4}{\pi^2} (1 - 1/4) \pi^2 / 6 = n^2 / 2$, where $\zeta^o(s) = \sum_{k \geq 1, k \text{ odd}} k^{-s}$. For the second, we obtain that the sum is equivalent to $\frac{16n^4}{\pi^4} \zeta^o(4) = n^4 \frac{16}{\pi^4} (1 - 1/16) \pi^4 / 90 = n^4 / 6$. \square

Lemma 17.

$$\sum_{k, l=1, k \text{ odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n}) / \sin^2(\frac{\pi l}{2n}) \right) \cot^2(\frac{\pi k}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} = O(n^4).$$

Proof. We use that there exists a constant $c > 0$ such that $cx \leq \sin(x) \leq x$ for all $x \in [0, \pi/2]$ to bound the summand by $(\frac{2n}{c\pi})^4 \frac{1}{l^2(k^2+l^2)}$. Since $\frac{1}{l^2(k^2+l^2)}$ is summable, we obtain a $O(n^4)$. \square

Lemma 18.

$$\sum_{k, l=1, l \text{ odd}}^{n-1} \frac{\left(\sin^2(\frac{\pi k}{2n}) / \sin^2(\frac{\pi l}{2n}) \right) \cot^2(\frac{\pi l}{2n})}{\sin^2(\frac{\pi k}{2n}) + \sin^2(\frac{\pi l}{2n})} = \frac{n^5}{6} + O(n^4).$$

Proof. Let $\varepsilon \in (0, 1)$ and $a \geq 1$. We decompose the sum in three contributions. The first one corresponds to $l \geq \varepsilon n$. This term gives a contribution of $\frac{1}{\varepsilon^6} O(\sum_{k=1}^{n-1} \sin^2(\frac{\pi k}{2n})) = O(n/\varepsilon^6)$.

We now write the remaining sum (for $l \leq \varepsilon n$) as the sum of two contributions: $k \leq al$ and $k \geq al$.

This first subcontribution gives a $O(a^2 n^4 \sum_{k, l \geq 1} \frac{1}{l^2(k^2+l^2)} (1 + O((1+a^2)\varepsilon^2)))$.

The second gives a contribution of

$$\left(\frac{2n}{\pi} \right)^4 \sum_{\substack{l \text{ odd}, l \leq \varepsilon n \\ k, k \geq al}} \frac{1}{l^4} (1 + O(\varepsilon^2)).$$

The first two terms are $O(n^4)$ and the last one becomes equivalent to

$$\begin{aligned} \left(\frac{2n}{\pi} \right)^4 (n-1) \sum_{l=1, l \text{ odd}}^{n/a} \frac{1}{l^4} &= \frac{16n^5}{\pi^4} (\zeta(4) - \frac{1}{2^4} \zeta(4)) + O(n^4 a) \\ &= \frac{n^5}{6} + O(n^4), \end{aligned}$$

since $\sum_n^\infty 1/l^4 = O(1/n)$. \square

5. OTHER DOMAINS

We consider a planar domain (closed, simply-connected) $D \subset \mathbb{C}$ with smooth boundary ∂D . For any $\varepsilon > 0$, we consider a finite graph D^ε whose vertices are a subset of $D \cap \varepsilon\mathbb{Z}^2$, where edges join points at distance ε , and which is “simply connected” in the sense that the union of its closed faces is simply connected. A sequence $(D^\varepsilon)_{\varepsilon>0}$ is said to *approximate* D if the Hausdorff distance from D^ε to D is $O(\varepsilon)$. In this section, we give asymptotics for the two-point probabilities $\mathbb{P}(f, f' \in \Gamma_\varepsilon)$ and the first and second moments of the area of the uniform cycle-rooted spanning tree Γ_ε on an approximating sequence $(D^\varepsilon)_{\varepsilon>0}$. The proofs are generalizations of the proofs of the previous section and somewhat technical so we only give sketches here. For complete proofs, see [5].

In this section, $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ is the positive definite Laplacian. Recall that the Green’s function with Neumann (resp. Dirichlet) boundary conditions is the (unique up to constant) smooth symmetric kernel over D solution to the PDE

$$\Delta_y u(x, y) = \delta_x,$$

with boundary condition $\frac{du(x, y)}{dn(y)} = 0$ for $y \in \partial D$, where $n(y)$ is the normal vector at the boundary point y (resp. $u(x, y) = 0$ for $y \in \partial D$). On a bounded domain the Neumann Green’s function is not defined in general; we can only invert Δ_y on functions of mean zero.

We denote g_D^r the Neumann Green’s function and g_D^0 the Dirichlet Green’s function. They both can be written in terms of the planar Brownian motion, respectively reflected or absorbed at the boundary.

A *rectilinear approximation* of D is an approximating sequence $(D^\varepsilon)_{\varepsilon>0}$ such that the boundaries are locally horizontal or vertical: any site in ∂D^ε belongs to a piece of horizontal or vertical segment of ∂D^ε with length $\delta = \delta(\varepsilon)$ such that $\varepsilon = o(\delta)$. We suppose that all our approximations are rectilinear.

We furthermore suppose that the domain is smooth enough so that g_D^r is at least twice differentiable.

Given a point $z \in D$ and an approximation $(D^\varepsilon)_{\varepsilon>0}$, we say that a sequence of vertices $v_\varepsilon \in D^\varepsilon$ approximates z if $|z - v_\varepsilon| = O(\varepsilon)$.

Let T^ε be the transfer impedance on D^ε .

Theorem 5. *Let $z_1 \neq z_2$ be two points of D , possibly on the boundary, and $v_1^\varepsilon, v_2^\varepsilon$ approximations in D^ε of z_1, z_2 respectively. Let $e_1^\varepsilon, e_2^\varepsilon$ be two oriented edges of D^ε with fixed directions e_1 and e_2 , independent of ε , and with starting vertices v_1^ε and v_2^ε . Then*

$$(9) \quad T^\varepsilon(e_{v_1}^\varepsilon, e_{v_2}^\varepsilon) = \varepsilon^2 \frac{\partial}{\partial e_1} \frac{\partial}{\partial e_2} g_D^r(z_1, z_2) + o(\varepsilon^2),$$

where for $i = 1$ and 2 , the symbol $\frac{\partial}{\partial e_i}$ denotes the partial directional derivative along direction e_i for the i -th variable.

If we restrict to the case where z_1 and z_2 are in the interior of D , then we actually don't need that D admit rectilinear approximations, and the proof works for any smooth domain.

In order to prove Theorem 5, we actually need its analog for $D = \mathbb{H}$ the upper half-plane. We will also use this result in the proof of Theorem 7. The proof is in fact easier in this case, see the sketch below.

The Green's function on \mathbb{C} has the explicit form $g_{\mathbb{C}}^0(x, y) = -\frac{1}{2\pi} \log |x - y|$. On the upper half-plane \mathbb{H} , the Neumann Green's function is given by $g_{\mathbb{H}}^r(x, y) = g_{\mathbb{C}}^0(x, y) + g_{\mathbb{C}}^0(\bar{x}, y)$.

Now, for any infinite planar graph \mathcal{G} with a (possibly empty) distinguished set of vertices called the boundary, the Neuman Green's function is defined in terms of the simple random walk as follows

$$G_{\mathcal{G}}^r(x, y) = \sum_{n=0}^{\infty} (\mathbb{P}(\text{SRW}_x(n) = y) - \mathbb{P}(\text{SRW}_x(n) = x)),$$

where $\text{SRW}_x(n)$ is the n -th position of a simple random walk on \mathcal{G} , started at x and reflected on the boundary.

Take $\mathbb{Z} \times \mathbb{N}$ to be the upper half-plane square lattice with boundary $\mathbb{Z} \times \{0\}$. Then we have $G_{\mathbb{Z} \times \mathbb{N}}(x, y) = G_{\mathbb{Z}^2}(x, y) + G_{\mathbb{Z}^2}(\bar{x}, y)$. The transfer impedance is defined in terms of the Green's function in the same way as for finite graphs. Note that $T_{\mathbb{Z}^2}$ is the limit of the transfer impedance on a sequence of finite subgraphs exhausting \mathbb{Z}^2 , see [1].

For $D = \mathbb{C}$ or $D = \mathbb{H}$ and for any $\varepsilon > 0$ we define D^ε to be $\varepsilon \mathcal{G}$ where $\mathcal{G} = \mathbb{Z}^2$ or $\mathcal{G} = \mathbb{Z} \times \mathbb{N}$, respectively.

Theorem 6. *Let $D = \mathbb{C}$ or \mathbb{H} . Let $z_1 \neq z_2$ be two points of D , possibly on the boundary, and $v_1^\varepsilon, v_2^\varepsilon$ approximations in D^ε of z_1, z_2 respectively. Let $e_1^\varepsilon, e_2^\varepsilon$ be two oriented edges of D^ε with fixed directions e_1 and e_2 , independent of ε , and with starting vertices v_1^ε and v_2^ε . Then*

$$(10) \quad T^\varepsilon(e_{v_1}^\varepsilon, e_{v_2}^\varepsilon) = \varepsilon^2 \frac{\partial}{\partial e_1} \frac{\partial}{\partial e_2} g_D^r(z_1, z_2) + o(\varepsilon^2),$$

where for $i = 1$ and 2 , the symbol $\frac{\partial}{\partial e_i}$ denotes the partial directional derivative along direction e_i for the i -th variable.

Sketch of proof of Theorem 6. We start by showing (10) in the case where D is the whole plane using the asymptotics

$$G_{\mathbb{Z}^2}(0, z) = -\frac{1}{2\pi} \log |z| + c + O\left(\frac{1}{|z|^2}\right),$$

for the Green's function on \mathbb{Z}^2 , see [11], knowledge of the error term (which extends to a continuous harmonic function), and the mean value property estimates (Proposition A.2 of [2]) to control the remainder terms.

We then deduce the formula for the half-plane by using a reflection principle, that is, writing that

$$G_{\mathbb{Z} \times \mathbb{N}}(u, v) = G_{\mathbb{Z}^2}(u, v) + G_{\mathbb{Z}^2}(\bar{u}, v),$$

where $G_{\mathbb{Z} \times \mathbb{N}}$ is the Green's function with Neumann boundary condition on the half-plane square lattice. \square

Sketch of proof of Theorem 5. Recall the definition $T(ab, uv) = G(a, u) - G(b, u) - G(a, v) + G(b, v)$ of the transfer impedance in terms of the Neumann Green's function. The proof follows by a careful study of the error term when replacing the discrete Green's function by g_D^r . In the case where both z_1 and z_2 are interior points, we may use the convergence results of discrete Green's functions to their continuous counterparts and the mean value property estimates [2] as follows:

Let D' be a compact simply connected subset of the interior of D containing both z_1 and z_2 . For any w_1 in D' , define

$$H^\varepsilon(w_1) := \frac{1}{\varepsilon^2} (T^\varepsilon(u_1^\varepsilon, e_2^\varepsilon) - T_{\varepsilon\mathbb{Z}^2}(u_1^\varepsilon, e_2^\varepsilon)),$$

where u_1^ε is the oriented edge with direction e_1 whose starting vertex is w_1 .

We now use the mean value property estimates for the variable w_1 to replace each term of H^ε by its average over a discrete ball of small radius $r > 0$. For fixed w_1 and r , the limit $\lim_{\varepsilon \rightarrow 0} H^\varepsilon(w_1)$ exists and is equal to

$$\frac{1}{\pi r^2} \left(\int_{\partial B(w_1, r)^+} \frac{\partial}{\partial e_2} h_D(w_1, z_2) ds - \int_{\partial B(w_1, r)^-} \frac{\partial}{\partial e_2} h_D(w_1, z_2) ds \right),$$

when $w_1 \in \partial D'$, where $h_D(w_1, w_2) = g_D^r(w_1, w_2) - g_C(w_1, w_2)$ and $\partial B(w_1, r)^+$ and $\partial B(w_1, r)^-$ are the right and left half-circles the union of which is the circle of radius r around w_1 .

By further taking $r \rightarrow 0$, we obtain that $\lim_{\varepsilon \rightarrow 0} H^\varepsilon(w_1) = \frac{\partial^2}{\partial e_1 \partial e_2} h_D(w_1, z_2)$ when $w_1 \in \partial D'$.

We now let g be the harmonic extension in D' of the function $\frac{\partial^2}{\partial e_1 \partial e_2} h_D(w_1, z_2)$ on $\partial D'$. It follows from classical results that for any $w_1 \in D'$ we have $H^\varepsilon(w_1, z_2) - g(w_1, z_2) = o(1)$. Since $\frac{1}{\varepsilon^2} T_{\varepsilon\mathbb{Z}^2}(u_1^\varepsilon, e_2^\varepsilon)$ converges to $\frac{\partial^2}{\partial e_1 \partial e_2} g_C(w_1, z_2)$ by Theorem 6, we then deduce that $\frac{1}{\varepsilon^2} T^\varepsilon(u_1^\varepsilon, e_2^\varepsilon)$ converges, as a function of w_1 , to a function which takes

boundary values $\frac{\partial^2}{\partial e_1 \partial e_2} g_D^r(w_1, z_2)$ and has a single second order pole with coefficient $-\frac{1}{2\pi}$. This uniquely determines its limit to be $\frac{\partial^2}{\partial e_1 \partial e_2} g_D^r(w_1, z_2)$ on the whole domain D' . In particular, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} T^\varepsilon(e_1^\varepsilon, e_2^\varepsilon) = \frac{\partial^2}{\partial e_1 \partial e_2} g_D^r(z_1, z_2).$$

In the case where at least one of the points z_1 or z_2 is on the boundary, we use a reflection argument (making full use of the rectilinearity of the domain), and use the result for interior points. \square

In a way, the previous theorem is a refinement of a result of [6] (Lemma 17), where the quantity $\frac{1}{\varepsilon}(G(v_1^\varepsilon + \varepsilon, v_2) - G(v_1^\varepsilon, v_2^\varepsilon))$ was shown to converge to the first partial derivative of g_D^0 .

We now suppose that D is bounded. Let κ_ε be the number of spanning trees of D^ε , and λ_ε its number of cycle-rooted spanning trees.

Theorem 7. *Let z_1 and z_2 be two distinct points in D , possibly on the boundary, and $v_1^\varepsilon, v_2^\varepsilon \in D^\varepsilon$ sequences of vertices approximating z_1, z_2 respectively. Let $f_{v_1^\varepsilon}$ and $f_{v_2^\varepsilon}$ be the faces whose lower left vertices are v_1^ε and v_2^ε . Then*

$$\mathbb{P}(f_{v_1^\varepsilon}, f_{v_2^\varepsilon} \in \Gamma_\varepsilon) = \frac{\kappa_\varepsilon}{\lambda_\varepsilon} (g_D^0(z_1, z_2) + o(1))$$

Sketch of proof. We do the proof in two steps.

First, we use Lemma 8 to write this probability as a sum over a product of two paths of edges. We let γ_1^ε and γ_2^ε be two rectilinear, non-crossing dual paths in D^ε from the unbounded face to $f_{v_1^\varepsilon}$ and $f_{v_2^\varepsilon}$, chosen in such a way that these paths approximate (with $O(\varepsilon)$ error) two non-crossing continuous paths γ_1 and γ_2 from ∂D to z_1 and z_2 . Lemma 8 states that

$$\mathbb{P}(f_{v_1^\varepsilon}, f_{v_2^\varepsilon} \in \Gamma_\varepsilon) = -\frac{\kappa_\varepsilon}{\lambda_\varepsilon} \sum_{e_1 \in E(\gamma_1^\varepsilon), e_2 \in E(\gamma_2^\varepsilon)} T^\varepsilon(e_1, e_2),$$

where $E(\gamma_1^\varepsilon)$ and $E(\gamma_2^\varepsilon)$ are the set of coherently oriented edges (let us say, counter-clockwise around $f_{v_1^\varepsilon}$ and $f_{v_2^\varepsilon}$) on the paths γ_1^ε and γ_2^ε . Using Theorem 5 we thus obtain, by a convergence argument, that

$$(11) \quad \frac{\lambda_\varepsilon}{\kappa_\varepsilon} \mathbb{P}(f_{v_1^\varepsilon}, f_{v_2^\varepsilon}) = - \int_{\gamma_1} \int_{\gamma_2} \frac{\partial}{\partial e_1} \frac{\partial}{\partial e_2} g_D^r(w_1, w_2) |dw_1| |dw_2| + o(1).$$

where for $i = 1, 2$, notation e_i denotes the normal direction to the right of path γ_i when followed from ∂D to z_i .

The second step is to evaluate this integral. We observe that this integral transforms covariantly under conformal transformation: if $\phi : D \rightarrow \mathbb{H}$ is a Riemann map

to the upper half-plane \mathbb{H} then

$$g_D^r(w_1, w_2) = -\frac{1}{2\pi} \log |\phi(w_1) - \phi(w_2)| - \frac{1}{2\pi} \log |\phi(w_1) - \overline{\phi(w_2)}|.$$

This is the real part of the function

$$\tilde{g}_D(w_1, w_2) = -\frac{1}{2\pi} \log(\phi(w_1) - \phi(w_2)) - \frac{1}{2\pi} \log(\phi(w_1) - \overline{\phi(w_2)}).$$

Moving the integration to the upper half-plane with $z_i = \phi(w_i)$, (11) becomes

$$\begin{aligned} &= -\frac{1}{2\pi} \operatorname{Re} \left(\int_{\phi(\gamma_1)} \int_{\phi(\gamma_2)} \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \log(z_1 - z_2) dz_1 dz_2 - \int_{\phi(\gamma_1)} \int_{\phi(\gamma_2)} \frac{\partial}{\partial z_1} \frac{\partial}{\partial \bar{z}_2} \log(z_1 - \bar{z}_2) dz_1 d\bar{z}_2 \right) \\ &= -\frac{1}{2\pi} \log \left| \frac{z_1 - z_2}{z_1 - \bar{z}_2} \right| = g_{\mathbb{H}}^0(z_1, z_2) = g_D^0(w_1, w_2), \end{aligned}$$

where we used the conformal invariance of g_D^0 in the last equality. \square

Let \mathbf{L}_ε and \mathbf{A}_ε be the combinatorial length and area of Γ_ε .

Theorem 8. *We have*

$$\mathbb{E}(\mathbf{L}_\varepsilon) = 8 + o(1),$$

$$\mathbb{E}(\mathbf{A}_\varepsilon) = -\frac{4}{\pi} \log \varepsilon + o(\log \varepsilon),$$

and

$$\mathbb{E}(\mathbf{A}_\varepsilon^2) = \varepsilon^{-2} C(D) |D| + o(\varepsilon^{-2}),$$

where $|D|$ is the Lebesgue measure of D and $C(D) = \frac{8}{|D|^2} \int_{D^2} g_D^0(z, w) |dz|^2 |dw|^2$ is the mean normalized exit time referred to in the Introduction.

Sketch of proof. The first equality is an easy consequence of a result of Levine and Peres [12]. The two others follow from the fact that moments are obtained as sums of probabilities as given in Lemma 3.

In order to evaluate the sums, we use that

$$\left| T^\varepsilon(e_1^\varepsilon, e_2^\varepsilon) - T_{\varepsilon\mathbb{Z} \times \varepsilon\mathbb{N}}(e_1^\varepsilon, e_2^\varepsilon) \right| \leq C_0 \varepsilon^2,$$

for some $C_0 > 0$, which we show by the mean value property estimates and a reflection argument near the boundary. This allows us to replace T^ε by $T_{\varepsilon\mathbb{Z} \times \varepsilon\mathbb{N}}$ with an error term of $O(\varepsilon^2)$.

The second equality follows by using this approximation and an explicit computation for $T_{\varepsilon\mathbb{Z} \times \varepsilon\mathbb{N}}$.

For the third equality, we show convergence of the sum $\sum_{f, f'} \mathbb{P}(f, f' \in \Gamma_\varepsilon)$ to an integral. This works without trouble for the contribution of pairs of faces that are macroscopically far apart and follows from Theorem 7. For the pairs of faces that are within distance $O(\varepsilon)$ of one another, we replace T^ε by $T_{\varepsilon\mathbb{Z} \times \varepsilon\mathbb{N}}$ if they are in the

bulk, and if they lie on the edge, we use the electric flow interpretation to show that their contribution is negligible to the first order. \square

The expected length and area do not depend on the shape of the domain. However, the mean normalized exit time (and thus the second moment of the area) does depend on this shape. Here are two examples.

- (1) Let D be a $1 \times \tau$ rectangle. We have $C(D) = \frac{512\beta(\tau)}{\pi^6}$, where

$$\beta(\tau) = \sum_{k,l=1, \text{ both odd}}^{\infty} \frac{\tau}{k^2 l^2 (k^2 + \tau^2 l^2)}.$$

This follows from Theorem 8 using the explicit form of the Dirichlet Green's function on $\mathcal{G}_{n,n\tau}$, the $n \times n\tau$ rectangle in the square lattice (computation omitted).

- (2) For the unit disk $C(D) = 1/\pi$: the Dirichlet Green's function is $-\frac{1}{2\pi} \log \left| \frac{z-w}{1-\bar{z}w} \right|$. Substituting $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ we get (using symmetry we assume $\theta_2 = 0$ and multiply the result by 2π)

$$C = -\frac{8}{|D|^2} \int_0^1 \int_0^1 \int_0^{2\pi} \log \left| \frac{r_1 e^{i\theta_1} - r_2}{1 - r_1 r_2 e^{i\theta_1}} \right| r_1 r_2 d\theta_1 dr_1 dr_2$$

and writing the log as the log of the numerator minus the log of the denominator gives

$$= -\frac{16}{\pi} \int_0^1 \int_0^1 (\log \max\{r_1, r_2\} - \log 1) r_1 r_2 dr_1 dr_2 = \frac{1}{\pi}.$$

Pólya proved that $C(D)$ (in a slightly different form) is maximized for the disk (and only the disk) and the argument goes as follows:

Let $\frac{1}{P(D)} = \inf_f w(f)$ be the infimum, over all smooth functions f over D vanishing on the boundary, of $w(f) := \frac{\int_D |\nabla f|^2}{4(\int_D f)^2}$. This defines a quantity $P(D)$, which Pólya called the *torsional rigidity* of the cross-section D , and which is, in mechanical terms, a measure of the resistance to torsion of a cylindrical beam with cross-section D . In fact the infimum of $w(f)$ is realized and we have $\frac{1}{P(D)} = w(f_0)$, where f_0 is the solution of $\Delta f = 2$ with Dirichlet boundary conditions. Since $f_0(z) = 2 \int_D g_D^0(z, w) |dw|^2$, we have

$$\begin{aligned} C(D) &= \frac{8}{|D|^2} \int_{D^2} g_D^0(z, w) |dz|^2 |dw|^2 = \frac{4}{|D|^2} \int_D f_0(w) |dw|^2 \\ &= \frac{4}{|D|^2} \frac{(\int_D f_0(w) |dw|^2)^2}{\int_D f_0(w) |dw|^2} = \frac{4}{|D|^2} \frac{(\int_D f_0(w) |dw|^2)^2}{\frac{1}{2} \int_D |\nabla f_0|^2 |dw|^2} = \frac{2}{|D|^2} P(D). \end{aligned}$$

In [14], Pólya proved an old conjecture of Saint-Venant (1856) that for fixed area, the disk is the domain that maximises $P(D)$. Therefore, since $C(D)$ is easily shown to be invariant under dilation, we have proved that $C(D)$ is maximized for the disk.

For the benefit of the reader here is a sketch of Pólya's proof which goes in two steps: He first shows that $C(D) \leq C(D')$ where D' is any Steiner symmetrization of D . (A Steiner symmetrization of D along line L is a set D' having L as symmetry axis and such that for any line L' perpendicular to L the set $L \cap D'$ is an interval of length equal to the length of the set $L' \cap D$.) He then uses a result of Gross (1917) (refined by Lyusternik) which states that for any domain D there is a countable sequence of Steiner symmetrizations whose limit is the disk with the same area as D .

6. HIGHER MOMENTS OF THE AREA

For any sequence (\mathcal{G}_n) such that $\frac{1}{n}\mathcal{G}_n$ approximates a domain D , we may write higher moments of the area of $\Gamma_{\mathcal{G}_n}$ as

$$\mathbb{E}(\mathbf{A}_{\mathcal{G}_n}^k) = \sum_{f_1, \dots, f_k} \mathbb{P}(f_1, \dots, f_k \in \Gamma_{\mathcal{G}_n}),$$

where the sum is over all ordered k -tuples of bounded faces of \mathcal{G}_n .

For any k fixed distinct faces f_1, \dots, f_k , we may try to compute the probabilities $\mathbb{P}(f_1, \dots, f_k \in \Gamma_{\mathcal{G}_n})$ by setting zippers from each of the f_i to the boundary (or equivalently, branch-cuts for the Green's function) with parallel transport q_i (or equivalently, monodromy q_i around faces f_i).

We conjecture the following.

Conjecture 1. *For any integer $k \geq 2$, there exists $a_k = a_k(D) > 0$ such that, when $n \rightarrow \infty$,*

$$\mathbb{E}(\mathbf{A}_{\mathcal{G}_n}^k) = a_k n^{2k-2} (1 + o(1)).$$

7. APPLICATION: SIZE OF FJORDS IN THE UNIFORM SPANNING TREE

The union of a spanning tree of a graph with any edge in its complement defines a unique cycle, which we call a *fjord* of that tree. We use the previous techniques to give a lower bound on the first and third moments of the length of a random fjord in a uniform spanning tree.

Let $\mathcal{G}_n \subset \mathbb{Z}^2$ be a sequence of graphs such that $\frac{1}{n}\mathcal{G}_n$ rectilinearly approximates a bounded domain $D \subset \mathbb{C}$. Consider the uniform spanning tree (UST) on \mathcal{G}_n . Add an independent random edge to the tree. Then one obtains a CRST with weight proportional to the length of its cycle. Using the previous results, we can obtain information on the size of this cycle. We denote by Υ_n the random CRST thus

obtained, by $\mathbb{E}_{\text{UST}_+}$ the corresponding expectation, and by \mathbf{L}_{Υ_n} and \mathbf{A}_{Υ_n} the length and area of its cycle.

Lemma 19. *For any measurable function f , we have*

$$\mathbb{E}_{\text{UST}_+}(f(\Upsilon_n)) = \frac{\mathbb{E}(f(\Gamma_n)\mathbf{L}_n)}{\mathbb{E}(\mathbf{L}_n)}.$$

Therefore, the results on the moments of the area of the uniform CRST proved earlier show that

Theorem 9.

$$\mathbb{E}_{\text{UST}_+}(\mathbf{A}_{\Upsilon_n}\mathbf{L}_{\Upsilon_n}^{-1}) = \frac{\log n}{2\pi} + o(\log n)$$

and

$$\mathbb{E}_{\text{UST}_+}(\mathbf{A}_{\Upsilon_n}^2\mathbf{L}_{\Upsilon_n}^{-1}) = \frac{C(D)|D|}{8}n^2 + o(n^2),$$

where $C(D)$ is the mean normalized exit time.

The isoperimetric inequality in the square grid reads $\mathbf{L}^2 \geq 16\mathbf{A}$. As a corollary, we thus obtain bounds on the first and third moments of \mathbf{L}_{Υ_n} .

Corollary 1.

$$\mathbb{E}_{\text{UST}_+}(\mathbf{L}_{\Upsilon_n}) \geq \frac{8}{\pi} \log n + o(\log n)$$

and

$$\mathbb{E}_{\text{UST}_+}(\mathbf{L}_{\Upsilon_n}^3) \geq 32C(D)|D|n^2 + o(n^2).$$

8. QUESTIONS

1. Can one do the previous computations on other lattices?
2. Here is a way to sample a CRST: suppose the edges of the graph are ordered; sample a uniform spanning tree (using David Wilson's very efficient algorithm) and add a random edge from the complement. This creates a cycle-rooted tree. If the edge added has minimal index in the set of edges belonging to the cycle thus closed, then output this cycle-rooted tree. Otherwise, throw this configuration away and sample again. This method works but is not very efficient. Is there an efficient way to sample uniform cycle-rooted trees?
3. What can be said about the distribution of the cycle of the uniform CRST on \mathbb{Z}^2 conditioned on surrounding the face at the origin?

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